A Quantum Algorithm for Minimum Steiner Tree Problem

Masayuki Miyamoto[1], Masakazu Iwamura[2], and Koichi Kise[2]

[1]Kyoto University, [2]Osaka Prefecture University

E-mail:miyamoto.masayuki.46s@st.kyoto-u.ac.jp / masa@cs.osakafu-u.ac.jp / kise@cs.osakafu-u.ac.jp

Introduction

Minimum Steiner Tree (MST) Problem

Well-known NP-hard problem

- ✓ **Input**: An undirected weighted graph $G=(V,E,w:E\to\mathbb{R}^+), |V|=n$, a terminal set $K\subseteq V, |K|=k$
- ✓ **Output**: A minimum Steiner tree T (minimum weighted tree that connects all vertices in the terminal set K)

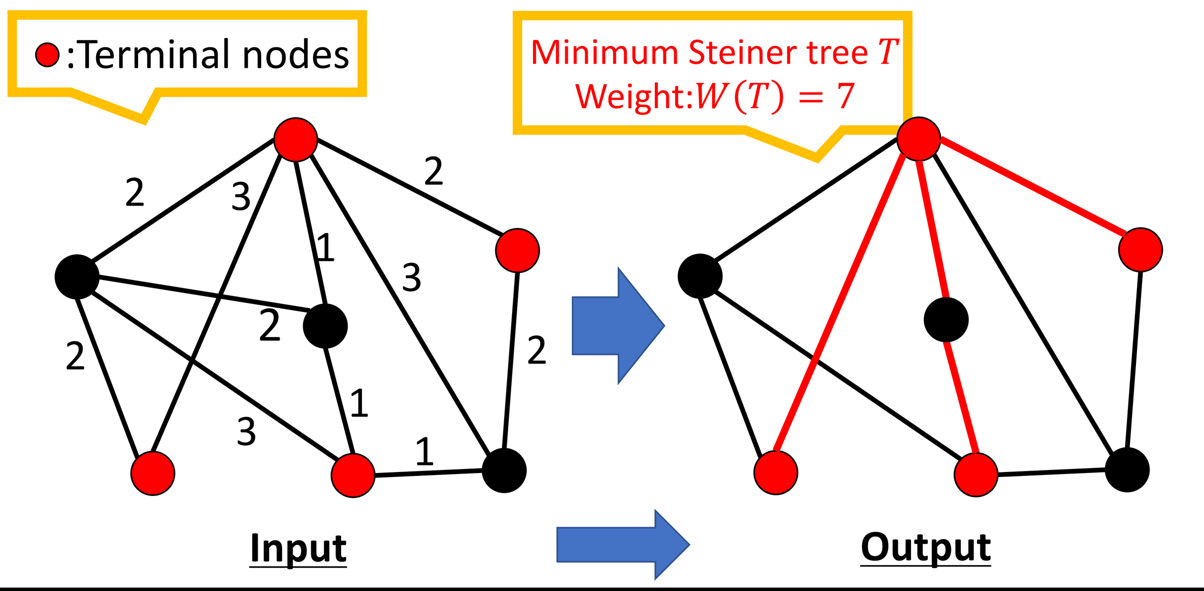


TABLE I. Comparison of the algorithms.

Algorithm	Complexity	classical or quantum
Dreyfus and Wagner [4]	$\mathcal{O}^*(3^k)$	classical
Fuchs [5]	$\mathcal{O}^*(2.684^k)$	classical
Mölle [6]	$\mathcal{O}((2+\delta)^k n^{f(\delta^{-1})})$	classical
Björklund [7] (best known in the restricted weight case)	$\mathcal{O}^*(2^k)$	classical
This paper	$\mathcal{O}^*(1.812^k)$	quantum

O^* notation hides a polynomial factor in n and k.

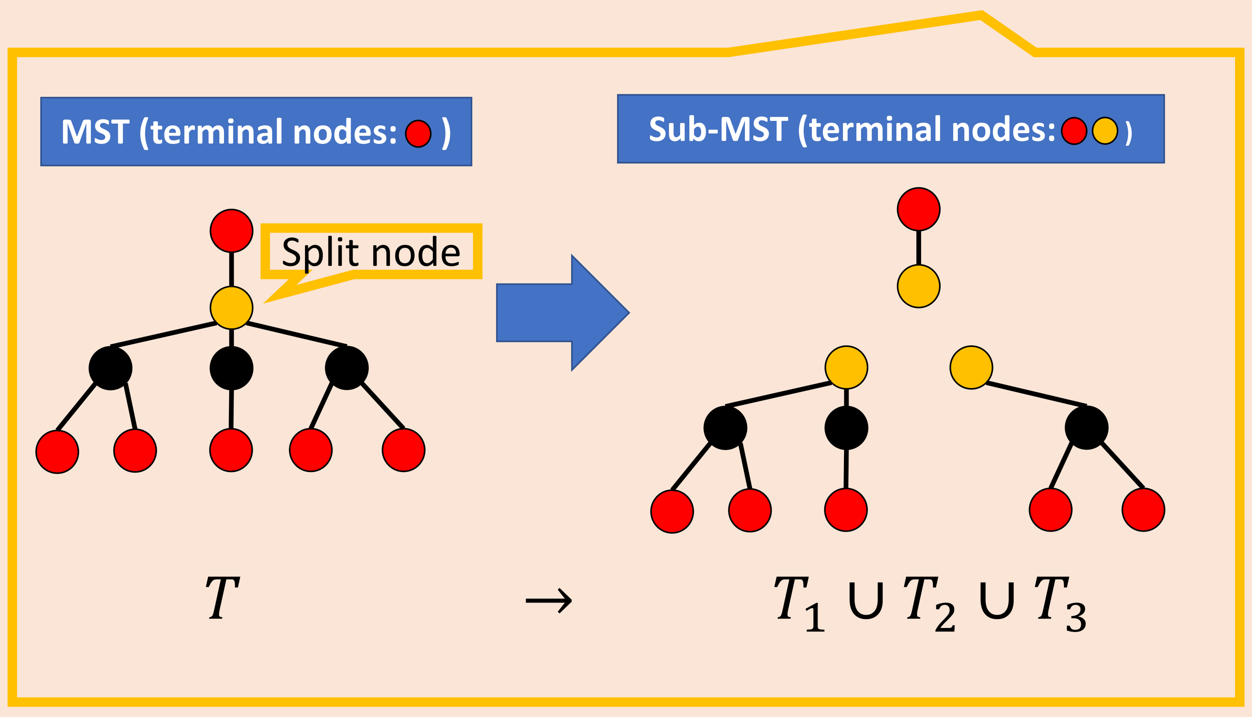
This Research

- ✓ proposed $O^*(1.812^k)$ algorithm
- ✓ No other algorithm faster than $O^*(2^k)$
- ✓ The first quantum algorithm for MST problem

Prior Work

Classical Algorithms for MST [DW71, FKW07]

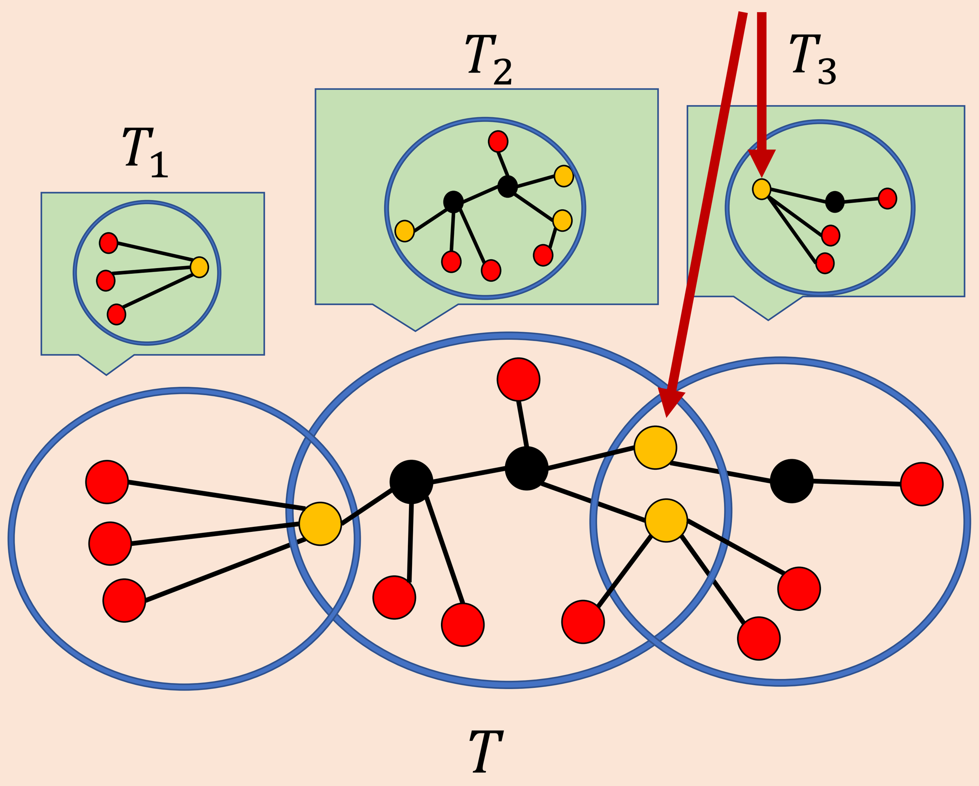
[DW71]: The tree decomposition into **three parts**



- ✓ The sizes of T_i are unknown
  → exhaustive search for all possible sizes
- ✓ Complexity: $O^*(3^k)$

[FKW07]: alternative decomposition into three parts

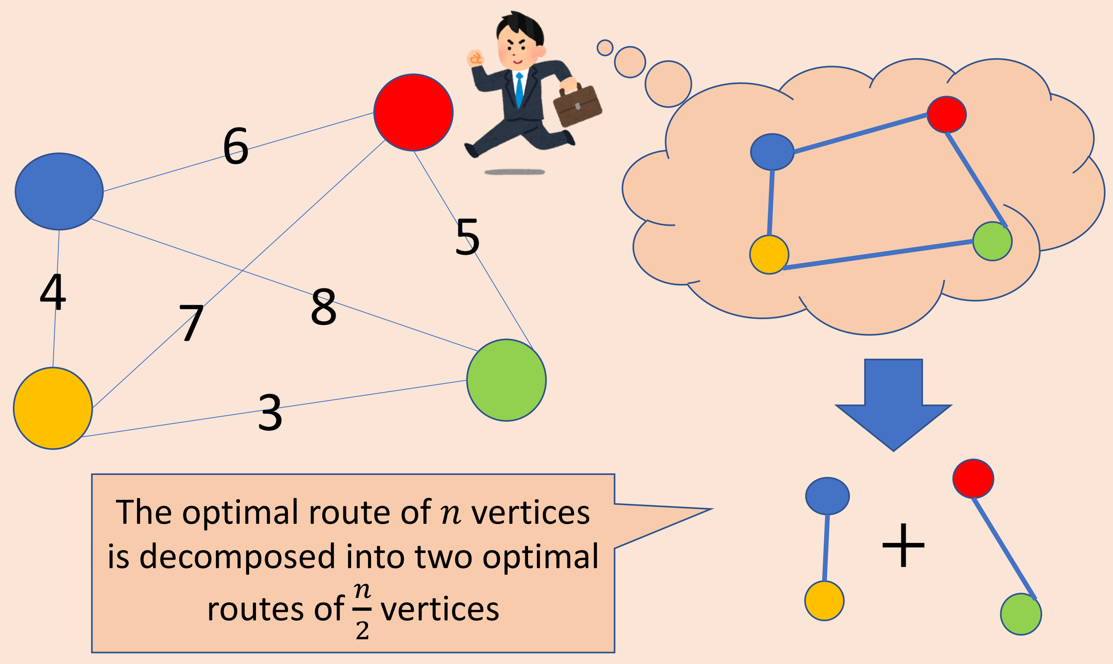
- ✓ Multiple split nodes
- ✓ Vertex contraction : **contract split nodes**



Quantum Algorithm for Travelling Salesman Problem [Amb+19]

Travelling Salesman Problem (TSP)

- ✓ **Input**: An undirected weighted graph $G=(V,E,w:E\to\mathbb{R}^+), |V|=n$
- ✓ **Output**: A minimum weighted route that visits each vertex



[Amb+19]

1. Classically compute the optimal routes of all possible half vertices
2. Quantumly compute **minimum value** in the two of these optimal routes

Finding minimum value in n elements	
quantum[DH96]	classical
$\tilde{O}(\sqrt{n})$	$O(n)$

The Proposed Algorithm

Relation to Prior Work

- ➢ Our algorithm has the structure similar to **[Amb+19]** (one classical part and one quantum part)
- ➢ Classical part : **[DW71]**
- ➢ Quantum part : **[FKW07]** (In quantum setting, two decomposition is better than three decomposition)

Algorithm

- ✓ Classical part : algorithm of **[DW71]**
- ✓ Quantum part : tree decomposition used in **[FKW07]**
  - ➢ The size of the set of split nodes A: $|A| = O(\log\frac{1}{\varepsilon})$
  - ➢ the MST is decomposed into **two sub-MSTs** whose sizes are $|K_1|=|K_2|=\left(\frac{1}{2}\pm\varepsilon\right)k$

$$W(K)=\min_{|K_1|=|V(T_1)\cap K|=\left(\frac{1}{2}\pm\varepsilon\right)k}\ \min_{A\subseteq V,|A|=\lceil\log_2\frac{1}{\varepsilon}\rceil}\{W(K_1\cup A)+W(K_2\cup v_A)_{G/A}\}\cdots(1)$$

Algorithm 2 Quantum algorithm for MINIMUM STEINER TREE

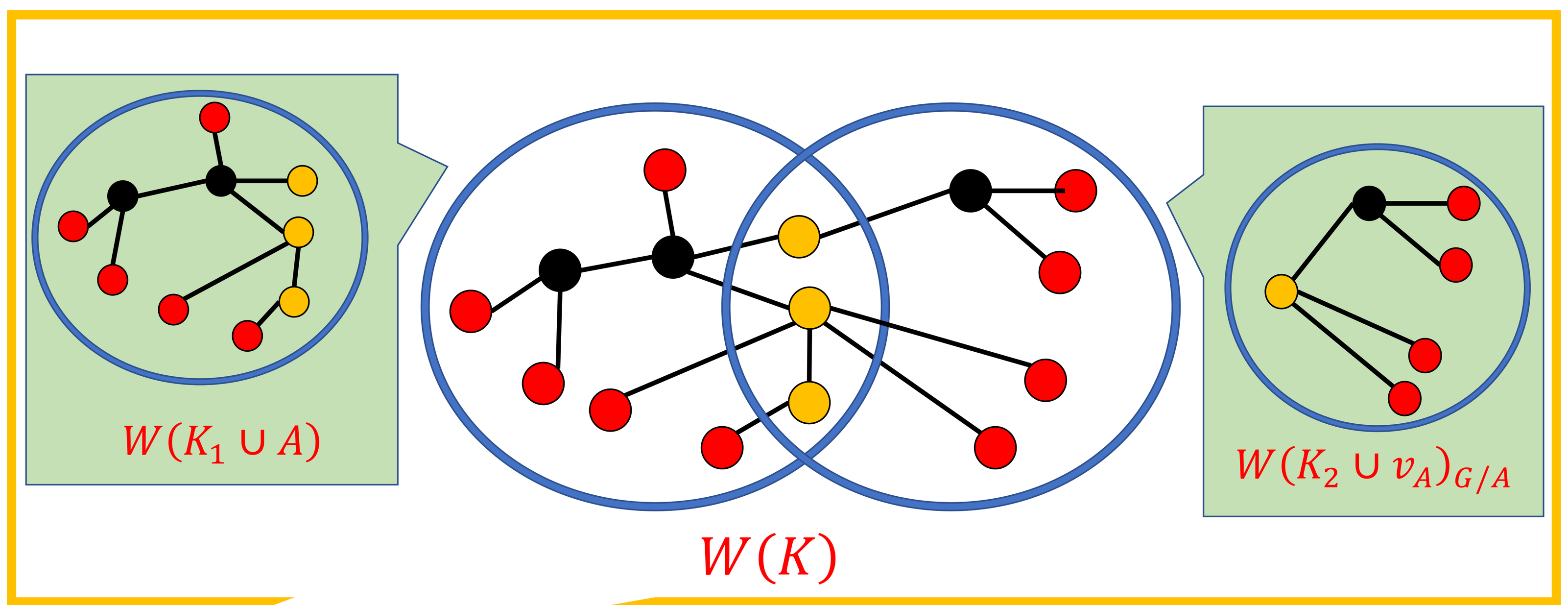
MinimumSteinerTree(graph $G=(V,E)$, edge weights w, a subset of vertices $K\subseteq V$): a minimum Steiner tree for K.

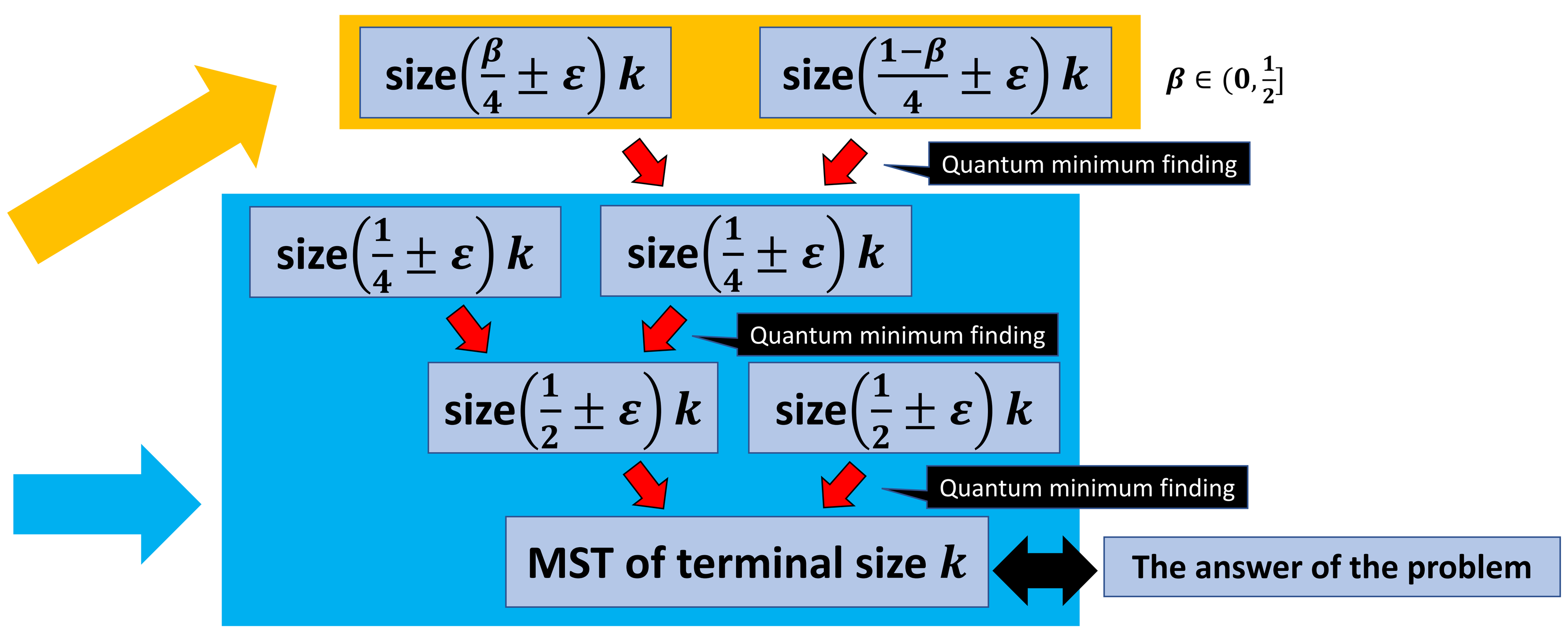
classical

1. Classically compute the values of $W(X)$ for all $X\subseteq K, |X|\le((1-\beta)/4+\varepsilon)k$ using dynamic programming.

quantum

2. (a) To calculate $W(K'')$ for $K''\subset K, |K''|=(\frac{1}{4}\pm\varepsilon)k$, apply D-H algorithm to Eq. (1) with $|K_1|=(\frac{\beta}{4}\pm\varepsilon)k$.
   (b) To calculate $W(K')$ for $K'\subset K, |K'|=(\frac{1}{2}\pm\varepsilon)k$, apply D-H algorithm to Eq. (1) with $|K_1|=(\frac{1}{4}\pm\varepsilon)k$.
   (c) Apply D-H algorithm to Eq. (1) with $|K_1|=(\frac{1}{2}\pm\varepsilon)k$ to find the solution.





Complexity

classical: $\mathcal{O}^*\left(\binom{(1+\varepsilon)k}{((1-\beta)/4+\varepsilon)k}2^{((1-\beta)/4+\varepsilon)k}\right)$

quantum: $\mathcal{O}^*\left(\sqrt{\binom{k}{(1+\varepsilon)k/2}\binom{(1+\varepsilon)k/2}{(1+\varepsilon)k/4}\binom{(1+\varepsilon)k/4}{(\beta+\varepsilon)k/4}}\right)$

$\beta\approx 0.2835,\ \varepsilon=0$ → $O^*(1.812^k)$

Classical setting

$\mathcal{O}^*\left(\binom{k}{(1+\varepsilon)k/2}\binom{(1+\varepsilon)k/2}{(1+\varepsilon)k/4}\binom{(1+\varepsilon)k/4}{(\beta+\varepsilon)k/4}\right)$

Some of References

- ✓ **[DW71]** : Stuart E Dreyfus and Robert A Wagner. The steiner problem in graphs. Networks, 1(3):195-207, 1971.
- ✓ **[FKW07]** : Bernhard Fuchs, Walter Kern, and Xinhui Wang. Speeding up the Dreyfus-Wagner algorithm for minimum Steiner trees. Mathematical methods of operations research, 66(1):117-125, 2007.
- ✓ **[Amb+19]** : Andris Ambainis, Kaspars Balodis, Jānis Iraids, Martins Kokainis, Krišjānis Prūsis, and Jevgēnijs Vihrovs. Quantum speedups for exponential-time dynamic programming algorithms. In Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 1783-1793. SIAM, 2019.
- ✓ **[DH96]** : Christoph Durr and Peter Hoyer. A quantum algorithm for finding the minimum. arxiv preprint quant-ph/9607014, 1996.